TRANSFORMATION GROUPS IN SPACE OF FOUR DIMENSIONS.

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In a previous article* a number of the most important definitions used in Lie's Theory of Groups were introduced. In this article those definitions are used without further explanation.

In another paper† the writer ventured, without proof, the assertion that he had found *all* the primitive groups in space of four dimensions. It is proposed to prove a part of that assertion by showing that *none of the groups which leave invariant two different manifoldnesses, each of two dimensions, in space of four dimensions, can be primitive.*

At the same time a most important part of the Theory of Groups will be illustrated by showing how groups which fulfil certain given conditions may be determined in the most general manner possible.

It will be necessary, in the first place, to introduce a few additional definitions.

By reference to the former paper, the reader will readily see that the symbol of an infinitesimal transformation in the four variables x_1, x_2, x_3, x_4 must have the form

$$Xf \equiv \xi_1(x_1, x_2, x_3, x_4)\frac{\partial f}{\partial x_1} + \xi_2(\,)\frac{\partial f}{\partial x_2} + \xi_3(\,)\frac{\partial f}{\partial x_3} + \xi_4(\,)\frac{\partial f}{\partial x_4} \equiv \sum_1^4{}_i\, \xi_i(\,)\frac{\partial f}{\partial x_i}\,.$$

Here the ξ_i's are, as usual, *analytical* functions of their arguments in Weierstrass's sense of the word; and hence, if $x_k^{(0)}$ is a point of ordinary position the ξ_i can be expanded in convergent powers of $(x_k - x_k^{(0)})$, at least for regions lying very near $x_k^{(0)}$. If this analytical expansion happens to begin with a term of the rth power, the infinitesimal transformation is said to be of the rth order. The first term of an expansion of this kind is called the *initial* term.

Two groups in n variables, each of r members, are said to be *similar*, when by a proper choice of the independent variables the one group can be transformed into the other. All groups, therefore, which are similar to a known group, may be considered known. Also, it has been proved by Lie that if a group can be assigned which has the same setting as a required

* Annals of Mathematics, Vol. VIII, No. 4, "On Transformation Groups."

† American Journal of Mathematics, Vol. X, No. 4, "On the Primitive Groups of Transformations in Space of Four Dimensions."

group, and if the initial terms of the respective transformations of the two groups are the same, then the two groups are *similar*. In practical calculations, therefore, only the initial terms of the transformations will be needed.

The introduction of certain properly chosen linear combinations of the transformations of a group, in place of the given transformations, often simplifies the "bracket operations" between the transformations of the group, by removing the indeterminate constants which may occur in the results of those operations. The performance of this simplification is called *normalizing* the transformations of the group.

Analogous to the definition of an *imprimitive* group in the plane, it is easy to see that a group in space of n dimensions is *imprimitive*, when it leaves invariant a family of ∞^{n-q} manifoldnesses, each of q dimensions, which fill the space of n dimensions exactly once.

The customary abbreviation p_i for the partial differential coefficients $\frac{\partial f}{\partial x_i}$ will be used.

I.—INTRODUCTORY.

1. *Transformation of zero and first order.*

If

$$x_1 = 0\,, \qquad x_2 = 0$$

and

$$x_3 = 0\,, \qquad x_4 = 0$$

are the two manifoldnesses which are invariant under the transformations of our group, it has been shown* that the transformations of the zero and of the first orders, respectively, of all possibly primitive groups, must have the following forms (writing, as usual, only the initial terms):

$$p_1,\ p_2,\ p_3,\ p_4\,;$$

$$x_1p_2\,,\ x_1p_1 - x_2p_2\,,\ x_2p_1\,;\quad x_3p_4\,,\ x_3p_3 - x_4p_4\,,\ x_4p_3\,; \tag{S_t}$$

$$x_1p_1 + x_2p_2 - x_3p_3 - x_4p_4\,; \tag{T}$$

with which transformations there *may* also occur

$$x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4 \equiv \sum_1^4{}_i\, x_ip_i\,. \tag{U}$$

2. *No transformation of an order higher than the second can occur.*

For, let s be the maximum order, so that a transformation of the order

$$Xf \equiv \sum_1^4{}_i\, \xi_i^{(s)}(x_1, x_2, x_3, x_4)\, p_i$$

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* American Journal of Mathematics, Vol. X, No. 4, p. 310.

occurs, where the index s shows that the $\xi_i^{(s)}$ begin with terms of the sth degree.

Since in all of the groups x_1, x_2 will be equally privileged with x_3, x_4, it may be assumed that $\xi_1^{(s)}$ and $\xi_2^{(s)}$ are not both zero. If now $\xi_1^{(s)}$ is not identically zero, it is seen, by properly combining Xf with p_1, p_2, p_3, and p_4 that

$$\xi_1^{(s)} \equiv \xi_1^{(s)}(x_1, x_2),$$

for the results of the bracket operations must always be capable of being linearly expressed in terms of the p_i, S_i, T, and U.

Furthermore, by combining Xf properly with $x_1 p_2$, it is seen that a transformation of the form

$$Yf \equiv x_1^{(s)} p_1 + y_2^{(s)} p_2 + y_3^{(s)} p_3 + y_4^{(s)} p_4$$

must occur in the group. Combine Yf and p_1; then

$$Zf \equiv s x_1^{(s-1)} p_1 + \bar{y}_2^{(s-1)} p_2 + \bar{y}_3^{(s-1)} p_3 + \bar{y}_4^{(s-1)} p_4$$

must occur. Now, combine Yf and Zf, and there results a transformation of the form

$$- s x_1^{2s-2} p_1 + \bar{\xi}_2^{(2s-2)} p_2 + (\,)\, p_3 + (\,)\, p_4 .$$

But the sth is the maximum order; hence

$$2s - 2 < s + 1 \quad \therefore \quad s < 3 .$$

Similarly if $\xi_1^{(s)} \equiv 0$ and $\xi_2^{(s)}$ be not 0, then also $s < 3$.

3. *To find the transformations of the second order.*

Let

$$\sum_1^4{}_i\, \xi_i^{(2)} p_i$$

be such a transformation.

If now $\xi_2^{(2)}$ and $\xi_4^{(2)}$ are not both identically zero, they can only be functions of x_3, x_4; and $\xi_1^{(2)}$, $\xi_2^{(2)}$ must contain only x_1, x_2.

(a) If $\xi_3^{(2)} \equiv \xi_4^{(2)} \equiv 0$, then $\xi^{(2)}$ and $\xi^{(2)}$ must evidently be both different from zero, and by the bracket operation it easily follows that the transformations of the second order are of the forms

$$Xf \equiv x_1 x_2 p_1 + x_2^2 p_2 , \quad Yf \equiv x_1^2 p_1 + x_1 x_2 p_2 .$$

(b) If $\xi_3^{(2)}$ and $\xi_4^{(2)}$ are not both identically zero, since the variables x_1, x_2 and x_3, x_4 are equally privileged, it may be assumed that also $\xi_1^{(2)}$ and $\xi_2^{(2)}$ are not zero. Therefore there are two transformations of the form

$$X_1 f \equiv x_1^2 p_1 + x_1 x_2 p_2 + \xi_3^{(2)}(x_3, x_4) p_3 + \xi_4^{(2)}(x_3, x_4) p_4 ,$$

and

$$Y_1 f \equiv x_1 x_2 p_1 + x_2^2 p_2 + y_3^{(2)}(x_3, x_4) p_3 + y_4^{(2)}(x_3, x_4) p_4 .$$

By combining $X_1 f$ with $x_1 p_2$, and $Y_1 f$ with $x_2 p_1$, it is seen that also in this case Xf and Yf must occur alone. In like manner

$$x_3 x_4 p_3 + x_4^2 p_4 , \quad x_3^2 p_3 + x_4 x_3 p_4$$

may occur alone; and these forms are all the possible transformations of the second order that can occur in this case.

II.—Case in which no transformations of the second order occur.

This case divides itself again according to whether the transformations

$$U \equiv \sum_1^4{}_i \, x_i p_i ,$$

and

$$T' \equiv x_1 p_1 + x_2 p_2 - x_3 p_3 - x_4 p_4$$

occur alone, or additively, i. e. in the form

$$a . U + \beta . T . \qquad (a, \beta = \text{const.})$$

1. *U and T occurring alone.*

The initial terms of the transformations are

$$\begin{gathered} p_1 , \ p_2 , \ p_3 , \ p_4 , \\ x_1 p_2 , \ x_1 p_1 - x_2 p_2 , \ x_2 p_1 ; \quad x_3 p_4 , \ x_3 p_3 - x_4 p_4 , \ x_4 p_3 \qquad (S_i) \\ U, \ T . \end{gathered}$$

Here we find at once the relations

$$\begin{gathered} (S_1, S_2) \equiv -2S_1 , \ (S_1, S_3) \equiv S_2 , \ (S_2, S_3) \equiv -2S_3 , \ (S_i, U) \equiv 0 , \\ (S_4, S_5) \equiv -2S_4 , \ (S_4, S_6) \equiv S_5 , \ (S_5, S_6) \equiv -2S_6 , \ (S_i, T) \equiv 0 ; \\ (U, T) \equiv 0 . \end{gathered}$$

We wish to normalize now with the transformation U, in order to find the setting of our group.

We know that

$$(p_1, U) \equiv p_2 + \sum_1^6{}_i \, a_i S_i + \beta U + \gamma T , \qquad (a_i, \beta, \gamma = \text{const.})$$

We may introduce a new p_1 by the substitution

$$\bar{p}_1 \equiv p_1 + \sum_1^6{}_i \, A_i S_i + B U + G T , \quad (A_i, B, G = \text{const.})$$

Then we find

$$(\bar{p}_1, U) \equiv p_1 + \sum_1^6{}_i\, a_i S_i + \beta U + \gamma T - \sum_1^6{}_i\, A_i S_i - BU - GT .$$

Since the A_i, B, and G are arbitrary constants, we can choose

$$A_i \equiv a_i\,, \quad B \equiv \beta\,, \quad G \equiv \gamma\,,$$

and, therefore,

$$(\bar{p}_1, U) \equiv \bar{p}_1\,;$$

or, as nothing depends upon the symbol we use, we can write

$$(p_1, U) \equiv p_1\,.$$

Thus p_1 and U are normally connected.

Proceeding analogously we can choose p_2, p_3, p_4 so that

$$(p_2, U) \equiv p_2\,, \quad (p_3, U) \equiv p_3\,, \quad (p_4, U) \equiv p_4\,.$$

Let us now find the relations between the p_k and the S_i and T. We have

$$(p_1, S_1) \equiv p_2 + \sum_1^6{}_i\, a_i S_i + \beta U + \gamma T\,, \qquad (a_i, \beta, \gamma = \text{const.})$$

Form Jacobi's identity by means of p_1, S_1, and U; thus,

$$((p_1, S_1), U) + ((S_1, U), p_1) + ((U_1, p_1), S_1) \equiv 0\,.$$

We find

$$(p_2, U) - (p_1, S_1) \equiv 0$$

or

$$(p_1, S_1) \equiv p_2\,.$$

This is a normal relation; and by proceeding analogously with all the transformations of the zero order, and those of the first order, we find that all the resulting relations are normal.

It only remains for us to find how the transformations of the zero order are connected among themselves.

We know that

$$(p_1, p_2) \equiv \sum_1^4{}_i\, a_i p_i + \sum_1^6{}_k\, \beta_k S_k + \gamma U + \delta T\,,$$

where a_i, β_k, γ, δ are certain constants.

Form Jacobi's identity with p_1, p_2, and U; thus,

$$((p_1, p_2), U) + ((p_2, U), p_1) + ((U, p_1), p_2) \equiv 0\,,$$

or

$$\sum_1^4{}_i\, a_i p_i - 2\,(p_1, p_2) \equiv 0\,,$$

or

$$(p_1, p_2) \equiv 0 .$$

In the same way we find that the rest of the (p_i, p_k)'s are normal relations.

This gives us the group

$$\boxed{\begin{array}{c} p_1, p_2, p_3, p_4\,;\ x_1 p_2, x_1 p_1 - x_2 p_2, x_2 p_1\,;\ x_3 p_4, x_3 p_3 - x_4 p_4, x_4 p_3\,; \\ x_1 p_1 + x_2 p_2 - x_3 p_3 - x_4 p_4,\ x_1 p_1 + x_2 p_2 + x_3 p_3 + x_4 p_4 . \end{array}}$$

But this group is *imprimitive*, since it contains two invariant subgroups, p_1, p_2 and p_3, p_4; that is, the two families of $\infty^2 M_2$: $\begin{cases} x_1 = c_1 \\ x_2 = c_2 \end{cases}$ and $\begin{cases} x_3 = c_3 \\ x_4 = c_4 \end{cases}$ (c_i const.) are invariant.

2. *U and T not occurring alone, but in the combined form* $W = \alpha U + \beta T$.

In this case, again, all the transformations of the first order are connected by normal relations. Let us normalize those of zero order by means of

$$S \equiv x_1 p_1 - x_2 p_2 + x_3 p_3 - x_4 p_4 ,$$

which is evidently a transformation of our group. As in (1), we can easily choose the $p_1, \ldots, p_4$ so that the following relations hold:

$$(p_1, S) \equiv p_1 , \quad (p_2, S) \equiv -p_2 \quad (p_3, S) \equiv p_3 , \quad (p_4, S) \equiv -p_4 .$$

We find now, without difficulty, by means of Jacobi's identity, that all the relations between the transformations of the zero and of the first orders are normal.

We find further, forming Jacobi's identity with

$$p_i, p_k, S\,; \text{ and } p_i; p_k, S_j,$$

the following relations :

$$(p_1, p_2) \equiv a \,.\, W + b \,.\, S$$

$$(p_1, p_3) \equiv (p_2, p_4) \equiv 0$$

$$(p_1, p_4) \equiv c \,.\, W + d \,.\, S$$

$$(p_2, p_4) \equiv e \,.\, W + f \,.\, S$$

$$(p_3, p_4) \equiv m \,.\, W + n \,.\, S$$

where $a, \ldots, n$ are certain constants.

Now form Jacobi's identity with p_1, p_2, W; thus,

$$((p_1, p_2), W) + ((p_2, W), p_1) + ((W, p_1), p_2) \equiv 0 ,$$

and hence

$$(\alpha + \beta)\, a \equiv (\alpha + \beta)\, b \equiv 0\,.$$

Proceeding similarly with p_1, p_4, W, we find

$$a\,.\,c = a\,.\,d = 0\,;$$

and similarly with the other p_i, p_k, W, we find

$$a\,.\,e = a\,.\,f = 0\,,$$

$$a\,(\alpha - \beta) = b\,, \quad e\,(\alpha + \beta) = f\,, \quad c\,(\alpha + \beta) = -\,d\,,$$

$$m\,(\alpha + \beta) = -\,n\,, \quad c\,(\alpha - \beta) = -\,d, \quad e\,(\alpha - \beta) = -\,f.$$

Hence, $$f = e\,.\,\beta = d = c\,.\,\beta = 0\,.$$

(A) *Suppose now* $\alpha \gtrless 0$, then $c = d = 0$. If also $\alpha \gtrless -\,\beta$ we have $a = b = 0$; and we see at a glance that p_1, p_2 must form an invariant sub-group; and the group interests us no longer, as it must be imprimitive.

If $\alpha = -\,\beta$, we have $m = n = 0$; and then p_3, p_4 form an invariant sub-group. Thus we have no possible primitive group when $\alpha \gtrless 0$.

(B) *Suppose* $\alpha = 0$. Then if $\beta \gtrless 0$, all our other constants are zero, and we find a group

$$\boxed{\begin{gathered} p_1, p_2, p_3, p_4\,;\ x_1 p_2, x_1 p_1 - x_2 p_2, x_2 p_1\,;\ x_3 p_4, x_3 p_3 - x_4 p_4, x_4 p_3\,; \\ T \equiv x_1 p_1 + x_2 p_2 - x_3 p_3 - x_4 p_4\,. \end{gathered}}$$

But this group is evidently imprimitive.

If $\beta = 0$, then no transformation W occurs at all; and we get the above group again without the transformation T. This group is also evidently imprimitive.

III.—Cases in which transformations of the second order can occur.

1. $\xi_3^{(2)}$ *and* $\xi_4^{(2)}$ *identically zero.*

Then the transformations of the second order have the forms,

$$Xf \equiv x_1 x_2 p_1 + x_2^2 p_2\,, \quad Xf \equiv x_1^2 p_1 + x_1 x_2 p_2\,.$$

We have to make another subdivision, now, according to whether

$$U \equiv \sum_1^4{}_i\, x_i p_i\,, \text{ and } T \equiv x_1 p_1 + x_3 p_3 - x_4 p_4 + x_2 p_2$$

occur free or not.

(a) If U and T occur free, we have, besides these transformations and Xf and Yf,

$$p_1, p_2, p_3, p_4\,;\; x_1 p_2,\, x_1 p_1 - x_2 p_2,\, x_2 p_1\,;\; x_3 p_4,\, x_3 p_3 - x_4 p_4,\, x_4 p_3\,. \qquad (S_i)$$

We shall now find how these transformations are connected.

We have

$$(S_1, U) \equiv aXf + \beta Yf\,, \qquad (a,\ \beta \text{ constants.})$$

Now introduce a new S_1 by means of

$$\bar{S}_1 \equiv S_1 - aXf - \beta Yf\,,$$

and we find

$$(\bar{S}_1, U) \equiv 0\,;$$

or, as we may write it,

$$(S_1, U) \equiv 0\,.$$

In like manner we may choose the other S_i so that

$$(S_i, U) \equiv 0\,. \qquad (i = 2, \ldots, 6)$$

By means of Jacobi's identity we can now show that all the transformations of the first order are connected by normal relations. Thus, for example, we know that

$$(S_1, S_2) \equiv aXf + bY - 2S\,,$$

and

$$((S_1, S_2)\,U) + ((S_2, U)\,S_1) + ((U, S_1)\,S_2) = 0\,;$$

that is,

$$a = b = 0\,.$$

We can easily choose the p_i's so that

$$(p_i, U) \equiv p_i\,;\; i = 1, \ldots, 4\,;$$

and we find from Jacobi's identity that the p_i's are connected by normal relations with the other transformations of the first and second orders as well as with each other. This gives us the (imprimitive) group,

$$\boxed{\begin{gathered} p_1, p_2, p_3, p_4\,;\; x_1 p_2,\, x_1 p_1 - x_2 p_2,\, x_2 p_1\,;\; x_3 p_4,\, x_3 p_3 - x_4 p_4,\, x_4 p_3\,; \\ x_1 p_1 + x_2 p_2 + x_3 p_3 + x_4 p_4,\, x_1 p_1 + x_2 p_2 - x_3 p_3 - x_4 p_4\,;\; x_1 x_2 p_1 + x_2^2 p_2, \\ x_1^2 p_1 + x_1 x_2 p_2\,. \end{gathered}}$$

(b). If the transformations U and T do not occur free they only occur in the form,

$$\alpha U + \beta T. \qquad (\alpha, \beta \text{ constants})$$

Our transformations are those which we have above designated as p_i, S_k, Xf, Yf, and $\alpha U + \beta T$.

Let us combine p_2 and Xf; thus,

$$(p_2, Xf) \equiv x_1 p_1 + 2x_2 p_2 .$$

Also,

$$(p_1, Yf) \equiv 2x_1 p_1 + x_2 p_2 .$$

By addition, we see that a transformation of the former $x_1 p_1 + x_2 p_2$ must occur. That is, we must have

$$x_1 p_1 + x_2 p_2 = \rho \,.\, S_2 + \sigma(\alpha U + \beta T); \qquad (\rho, \sigma \text{ constants})$$

and hence

$$\rho = 0, \quad \alpha = \beta, \quad \sigma = 1 .$$

The transformation $\alpha U + \beta T$ has therefore the form,

$$x_1 p_1 + x_2 p_2 .$$

Let us normalize our relations, now, with the transformation

$$S \equiv x_1 p_1 + x_2 p_2 + x_3 p_3 - x_4 p_4 .$$

We notice that

$$(Xf, S) \equiv - Xf, \quad (Yf, S) \equiv - Yf;$$

and we can easily choose the S_k such that

$$(S_1, S) \equiv 0, \quad (S_2, S) \equiv 0, \quad (S_3, S) \equiv 0,$$

$$(S_4, S) \equiv - 2S_4, \quad (S_5, S) \equiv 0, \quad (S_6, S) \equiv 2S_6 .$$

It is easy to see that all the (S_i, S_k) are normal; and we can choose the p_i, in the usual manner, so that,

$$(p_1, S) \equiv p_1, \quad (p_2, S) \equiv p_2, \quad (p_3, S) \equiv p_3,$$

$$(p_4, S) \equiv - p_4 + aXf + bYf. \qquad (a, b \text{ constants})$$

We have

$$(p_4, S_2) \equiv \sum_1^6{}_i\, a_i S_i + \alpha T + \beta_2 Xf + \gamma_2 Yf;$$

and by Jacobi's identity,

$$- 2a_4 S_4 + 2a_6 S_6 - \beta_2 Xf - \gamma_2 Yf + \sum_1^6{}_i\, a_i S_i + \alpha T$$
$$+ \beta_2 Xf + \gamma_2 Yf + aXf + bYf \equiv 0,$$

or

$$a_i = a = b = a = 0\,.$$

Hence

$$(p_4, S) \equiv -p_4\,.$$

We find, by proceeding analogously, that all

$$(p_i, S_k) \qquad (i, k = 1, 2, 3)$$

are normal; as are also (p_i, Xf) and (p_i, Yf), $i = 1, \ldots, 4$.

Further,

$$(p_1, S_5) \equiv (p_1, S_6) \equiv (p_2, S_5) \equiv (p_2, S_6) \equiv (p_3, S_6) \equiv (p_4, S_4) = 0\,,$$

and

$$(p_3, S_5) \equiv p_3\,, \quad (p_4, S_6) \equiv p_3\,;$$

Also,

$$(p_1, p_2) \equiv (p_1, p_3) \equiv (p_2, p_3) \equiv 0\,.$$

By repeated applications of Jacobi's identity, we find moreover,

$$\begin{cases} (p_4, S_2) \equiv \beta_2 Xf + \gamma_2 Yf\,, \\ (p_4, S_1) \equiv \beta_1 Xf + \gamma_1 Yf\,, \\ (p_4, S_3) \equiv \beta_3 Xf + \gamma_3 Yf\,; \end{cases}$$

$$(\beta_1, \ldots, \gamma_3 \text{ constants})$$

and

$$\begin{cases} (p_1, S_4) \equiv a_1 Xf + b_1 Yf\,, \\ (p_2, S_4) \equiv a_2 Xf + b_2 Yf\,, \\ (p_3, S_4) \equiv p_4 + a_3 Xf + b_3 Yf\,, \\ (p_4, S_5) \equiv -p_4 + a_4 Xf + b_4 Yf\,. \end{cases}$$

$$(a_1, \ldots, b_4 \text{ constants})$$

Analogously, we find,

$$\begin{cases} (p_1, p_4) \equiv A_1 S_1 + A_2 S_2 + A_3 S_3 + A_5 S_5 + AS\,, \\ (p_2, p_4) \equiv B_1 S_1 + B_2 S_2 + B_3 S_3 + B_5 S_5 + BS\,, \\ (p_3, p_4) \equiv C_1 S_1 + C_2 S_2 + C_3 S_3 + C_5 S_5 + CS\,. \end{cases}$$

If, now, we form all Jacobian identities possible with the p_i, the S_k, S, Xf, and Yf, we will find, after a computation which we omit here on account of

its length,

$$\begin{cases} (p_1, p_4) \equiv -2B_2S_2\,, \\ (p_2, p_4) \equiv B_2S_2 - 3B_2(S_2, S_5)\,, \\ (p_4, S_1) \equiv 2B_2Xf\,, \\ (p_4, S_2) \equiv -2B_2Yf\,, \\ (p_4, S_3) \equiv (p_1, S_4) \equiv (p_2, S_4) \equiv 0\,, \\ (p_3, S_4) \equiv p_4 + 2B_2Yf\,, \\ (p_4, S_5) \equiv -p_4 - 2B_2Yf\,. \end{cases}$$

Let us now introduce a new p_4 by means of the substitution

$$\bar{p}_4 \equiv p_4 + 2B_2Yf\,;$$

and, as this does not affect those transformations which have already been normalized, we see that this is equivalent to making $B_2 = 0$.

All our relations are now normal, and we find thus the group

$$\boxed{\begin{gathered} p_1, p_2, p_3, p_4\,;\ x_1p_2, x_1p_1 - x_2p_2, x_2p_1\,;\ x_3p_4, x_3p_3 - x_4p_4, x_4p_3\,; \\ x_1p_1 + x_2p_2, x_1x_2p_1 + x_2^2p_2, x_1^2p_1 + x_1x_2p_2\,. \end{gathered}}$$

It is easy to see, however, that this group is imprimitive.

2. *Not all the* $\xi_3^{(2)}$ *and* $\xi_4^{(2)}$ *identically zero.*

Since the variables x_1, x_2 are equally privileged with x_3, x_4, we can also assume that not all $\xi_1^{(2)}$ and $\xi_2^{(2)}$ are identically zero. Thus we easily see that among the transformations of the second order must occur

$$X_1f \equiv x_1x_2p_1 + x_2^2p_2 + \xi_3^{(2)}(x_3, x_4)\,p_3 + \xi_4^{(2)}(x_3x_4)\,p_4\,,$$

and

$$X_2f \equiv x_1^2p_1 + x_1x_2p_2 + \eta_3^{(2)}(x_3x_4)\,p_3 + \eta_4^{(2)}(x_3, x_4)\,p_4\,.$$

Now combine X_1f and x_1p_2; hence

$$X_1f \equiv x_1^2p_1 + x_1x_2p_2$$

must occur alone; so also must evidently

$$X_2f \equiv x_1x_2p_1 + x_2^2p_2\,.$$

Then, since the variables are equally privileged, we see that

$$X_4f \equiv x_3x_4p_3 + x_4^2p_4\,,$$

and

$$X_3f \equiv x_3^2 p_3 + x_3 x_4 p_4$$

must also occur alone ; and these are all possible transformations of the second order.

If now, with the p_i, the S_k, and the $X_j f$,

$$U \equiv \sum_1^4 x_i p \text{ and } x_1 p_1 + x_2 p_2 - x_3 p_3 - x_4 p_4 \equiv T$$

occur alone, we easily find the (imprimitive) group

$$\boxed{\begin{array}{c} p_1, p_2, p_3, p_4 ; x_1 p_2, x_1 p_1 - x_2 p_2, x_2 p_1 ; x_3 p_4, x_3 p_3 - x_4 p_4, x_4 p_3 ; \\ x_1 p_1 + x_2 p_2 + x_3 p_3 + x_4 p_4 , \; x_1 p_1 + x_2 p_2 - x_3 p_3 - x_4 p_4 ; \\ x_1^2 p_1 + x_1 x_2 p_2, x_1 x_2 p_1 + x_2^2 p_2 ; x_3^2 p_3 + x_3 x_4 p_4, x_3 x_4 p_3 + x_4^2 p_4 . \end{array}}$$

If, however, U and T only occur in the combination

$$\alpha U + \beta T, \qquad (\alpha, \beta \text{ constants})$$

our transformations will not form a group at all. For

$$\tfrac{1}{3} \{ (p_2, X_2 f) + (p_1, X_1 f) \} \equiv x_1 p_1 + x_2 p_2 ,$$

or

$$x_1 p_1 + x_2 p_2 \equiv \rho (x_1 p_1 - x_2 p_2) + \sigma (x_3 p_3 - x_4 p_4) + \nu (\alpha U + \beta T). \qquad (\rho, \sigma, \nu \text{ constants})$$

Thus we see

$$\rho \equiv \sigma \equiv 0 ,$$

$$\nu (\alpha - \beta) \equiv 0 .$$

In order to have a group, therefore, we must have $\alpha = \beta$, and

$$\alpha U + \beta T \equiv x_1 p_1 + x_2 p_2 .$$

But

$$\tfrac{1}{3} \{ (p_4, X_4 f) + (p_3, X_3 f) \} \equiv x_3 p_3 + x_4 p_4$$

must belong to our group, if it is a group ; and we see at once that this transformation cannot have the necessary form

$$\sum_1^6 {}_i \, a_i S_i + a (x_1 p_1 + x_2 p_2) . \qquad (a_i, a \text{ constants})$$

Therefore, in this case, our transformations do not form any group.

Hence we have shown that there are no primitive groups of infinitesimal transformations in space of four dimensions which leave two different manifoldnesses of two dimensions invariant.